Alpha Relay Solutions

A_1 6 By Descartes' Rule of Signs, there are at most 2 positive roots.
$f(-x) = x^{10} + ax^8 + bx^6 + cx^5 - dx^2 + 360$. So there are at most 2 negative roots. As a result, there must be at least 6 imaginary roots.

A_2 76 There are 3 possible constant terms in the expansion

$$\frac{6!}{2!\,0!\,4!}(x^2)^2(1)^0\left(\frac{1}{x}\right)^4 + \frac{6!}{1!\,3!\,2!}(x^2)^1(1)^3\left(\frac{1}{x}\right)^2 + \frac{6!}{0!\,6!\,0!}(x^2)^0(1)^6\left(\frac{1}{x}\right)^0 = 76$$

A_3 8 Let x, y be the number of chickens and pigs, respectively. Then $x + y = 30$ and $2x + 4y = 76$. Solving to get $y = 8$.

A_4 27 $\frac{S_2}{S_1} = r^8$, so $r = \pm 2^{\frac{5}{4}}$. So the next integer in the sequence is $a_5 = 32$, regardless whether the common ratio is positive or negative. $32 - 5 = 27$.

A_5 13 Multiplying both sides by $27xy$ to get $27x + 27y = xy$, collecting to one side, $xy - 27x - 27y = 0$, adding $27^2 = 3^6$ to both sides and factor, we have $(x - 27)(y - 27) = 3^6$. For x, y to be integers, $(x - 27, y - 27)$must be of the form $(\pm 3^k, \pm 3^{6-k})$ for $k = 0, 1, \dots, 6$. However, since x, y are denominators, $(x - 27, y - 27)$ cannot be $(-27, -27)$, leaving 13 ordered pairs.

A_6 96 We can divide the Cartesian plane into 4 sections, based on when each absolute value expression is equal to 0. Specifically, using lines $x = 2y$ and $x = -y$. Within each of the 4 regions, the equation is linear, so we just need to find where the graph is on those 2 lines, which would be the 4 vertices of the quadrilateral.
On $x = 2y$, we have $|3y| = 12$, $y = \pm 4$, this correspond to $(8, 4), (-8, -4)$.
On $x = -y$, we have $|-3y| = 12$, $y = \pm 4$. This correspond to $(-4, 4), (4, -4)$.
The result is a parallelogram with base of 12 and height of 8. The area is 96.

B_1 5 Divide the equilateral triangle into 4 smaller ones by drawing its midsegments. Divide the hexagon into 6 triangles by drawing its 3 diagonals that connect opposite vertices. So the ratio of the areas is $\frac{2}{3}$, for a sum of 5. (or $\frac{3}{2}$, but the sum is the same)

B_2 5 There are 3 cases. First is where AB is the base, for which there is one possibility for C. Second is where A is the vertex, for which there are two possibilities for C, one resulting in an acute triangle, and one resulting in an obtuse triangle. Third case is where B is the vertex, this is the same as case 2, so two possibilities as well. There are a total of 5.

B_3 50 Call the center of the circles O, and let P be the midpoint of EF. Let $OP = x$, the radius of the large circle be R, and the radius of the small circle be r.
On ΔOPE, we have $x^2 + 7.5^2 = R^2$, on ΔOPG, we have $x^2 + 2.5^2 = r^2$.
Subtracting the 2 equations, we have $R^2 - r^2 = 7.5^2 - 2.5^2 = 50$. So the area of the annulus is 50π.

B_4	9	Let P be the point on XZ such that $YP \perp XZ$, then $YP = 50\sin\angle X = 40$. ΔXYZ exists if $YZ \geq YP = 40$, and ΔXYZ is not unique if YZ is on $(40, 50)$, so there are 9 possibilities.
B_5	28	Extend NM and PQ to intersect at point O, then ΔNOP is equilateral. Consider ΔMOQ, we have $MO = 6, QO = 4, \angle O = 60°$. Applying law of cosine, we have $MQ^2 = 6^2 + 4^2 - 2 \cdot 6 \cdot 4 \cdot \cos 60° = 28$.
B_6	12	Let the radius be r and the central angle be θ (in radians). Then $2r + r\theta = 24$ and $\frac{r^2\theta}{2} = 28$. From the second equation, we have $\theta = \frac{56}{r^2}$. Substitute into the first equation, $2r + \frac{56}{r} = 24$, or $r^2 - 12r + 28 = 0$. Solving using the quadratic formula gives $r = 6 \pm 2\sqrt{2}$, and both values give $\theta < 2\pi$, so the sum is 12.
C_1	15	$35 + 30 - 50 = 15$
C_2	11	The probability of drawing two red socks is $\frac{C_2}{15} \cdot \frac{C_2 - 1}{14}$. $C_2 = 11$ is the closest to $\frac{1}{2}$.
C_3	17	Let p be the probability of flipping heads once. Then $\binom{11}{4} p^4(1-p)^7 = \binom{11}{5} p^5(1-p)^6$, or $1 - p = \frac{7}{5}p$. Solving to get $p = \frac{5}{12}$, and $5 + 12 = 17$.
C_4	8	The longest side must be 6, 7, or 8. It is simple enough to just list the possibilities: $(8,8,1), (8,7,2), (8,6,3), (8,5,4), (7,7,3), (7,6,4), (7,5,5), (6,5,5)$.
C_5	96	To seat the 4 couples around the table, we'll first pretend each couple is one "person". There are 3! Ways to arrange the couples. The two people in each couple can switch their seats, so there are $3! \cdot 2^4 = 96$ ways for them to seat.
C_6	28	Let x, y be the number of 2-step and 3-step strides. Then $2x + 3y = 15$. There are a few possibilities for (x, y): $(0, 5)$, where there are $\binom{5}{0} = 1$ way to take those 5 steps; $(3, 3)$, where there are $\binom{6}{3} = 20$ ways to take those 6 steps; $(6, 1)$, where there are $\binom{7}{1}$ ways to take those 7 steps. Total is 28.
D_1	30	The period of the first half is 6, and the period of the second half is 10. They coincide at their least common multiple, which is 30.
D_2	60	Since 30 is even, there are 60 petals.
D_3	8	$\frac{60ft}{1min} \cdot \frac{12in}{1ft} \cdot \frac{1rad}{1.5in} \cdot \frac{1min}{60s} = \frac{8rad}{s}$

D_4 8 $5(1-\cos^2 x) = 8\cos x + 1$, so $5\cos^2 x + 8\cos x - 4 = 0$. Factor to get $(5\cos x - 2)(\cos x + 2) = 0$. So $\cos x = \frac{2}{5}$. On $[0, 4\pi)$, there are 4 solutions: $\alpha, 2\pi - \alpha, 2\pi + \alpha, 4\pi - \alpha$, where $\alpha = \cos^{-1}\left(\frac{2}{5}\right)$. So the sum is 8π.

D_5 47 $(\sin x + \cos x)^2 = 1 + 2\sin x\cos x = \left(\frac{9}{8}\right)^2 = \frac{81}{64}$, so $2\sin x\cos x = \frac{17}{64}$. Therefore, $(\sin x - \cos x)^2 = 1 - 2\sin x\cos x = \frac{47}{64}$.

D_6 8 We have $\cos(4x) = \sin(7x)$. There are two possibilities:
First $4x + 7x = \frac{\pi}{2} + 2\pi k$, or $x = \frac{(1+4k)\pi}{22}$. On $[0, \pi)$, $k = 0, 1, \dots, 5$.
Second: $7x - 4x = \frac{\pi}{2} + 2\pi k$, or $x = \frac{(1+4k)\pi}{6}$, On $[0, \pi)$, $k = 0, 1$.
There are no overlaps between the two cases, so there are $6 + 2 = 8$ solutions.

E_1 2 For n to be divisible by 99, it must be divisible by 9 and 11.
For 9, $X + 1 + 6 + 9 + Y + 9 \equiv 0\ (mod\ 9)$, or $X + Y \equiv 2\ (mod\ 9)$.
For 11, $X + 6 + Y \equiv 1 + 9 + 9\ (mod\ 11)$, or $X + Y \equiv 2\ (mod\ 11)$.
Therefore, $X + Y = 2$. For the larger value of n, $X = 2$.

E_2 50 $\det(A) = 3 \cdot 7 - 2 \cdot (-2) = 25$, $\det(B) = 3 \cdot 6 - 4 \cdot 4 = 2$.
So $\det(2AB^{-1}) = 2^2 \cdot 25 \div 2 = 50$.

E_3 20 Let $z = a + bi$, then $a^2 + b^2 = 50^2$. There are 3 possibilities for a and b. They can be 0 and 50, where there are 4 possibilities, accounting for swapping the two coordinates and negations. They can be 30 and 40, for 8 possibilities. Finally, they can be 14 and 48, for 8 possibilities. There are a total of 20 possibilities.

E_4 15 Let the angle between $5\sqrt{2}$ and $3\sqrt{10}$ be θ. Then $50 + 90 - 30\sqrt{20}\cos\theta = 20$. So $\cos\theta = \frac{4}{\sqrt{20}}$, and $\sin\theta = \frac{2}{\sqrt{20}}$. Then the area is $\frac{1}{2} \cdot 5\sqrt{2} \cdot 3\sqrt{10} \cdot \frac{2}{\sqrt{20}} = 15$.

E_5 54 The graph is an ellipse with eccentricity is $\frac{1}{2}$ with the major axis on the x-axis. Substitute in $0, \pi$ for θ to find the points $(5, 0), (15, \pi)$ as end points of the major axis and results in $a = 10$. So $c = 5, b = 5\sqrt{3}$, and the area is $\frac{50\pi\sqrt{3}}{1}$. Therefore, $m + n + q = 54$.

E_6 64 We have to consider the power modulus 8 and 9. Clearly, 20^{54} is divisible by 8. $20^{54} \equiv 2^{54} \equiv (2^6)^9 \equiv 1^9 \equiv 1\ (mod\ 9)$. So the remainder is the multiple of 8 between 0 and 71 that is 1 mod 9, which is 64.